

A Monte Carlo Study of Pairwise Comparison

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1 Basics of Pairwise Comparison Method

Making comparative judgments of intangible stimuli or criteria (e.g. the degree of an environmental hazard or pollution factors) involves not only imprecise or inexact knowledge but also inconsistency in our own judgements. The improvement of knowledge elicitation by controlling the inconsistency of experts' judgments is not only desirable but absolutely necessary. Due to space limitations, the reader's familiarity with [3] is assumed although this paper is not a continuation of [3] but addresses different aspects of the same theory. Only the essential concepts of the pairwise comparison method are presented here.

The basic model of knowledge engineering is based on teamwork in which a knowledge engineer mediates between human experts and the knowledge base. The knowledge engineer elicits knowledge from the experts, refines it with the experts, and represents it in the knowledge base. Arrow's impossibility theorem states that no solution to the problem of a group ranking exists under general assumptions (see [1, 4]); however, a constructive algorithm exists under modified (but still practical) assumptions when we are able to compare the stimuli in pairs.

The pairwise comparison methodology introduced by Thurstone in 1927 (see [12]) can be used as a powerful inference tool and knowledge acquisition technique in knowledge-based systems. Some of the notable applications are related to projects of national importance, e.g., decisions on nuclear power plants in Holland ([8]) and a transportation system in Sudan([10]).

The practical and theoretical virtue of the pairwise comparison methodology is its simplicity. The goal of pairwise comparisons is to establish the relative preferences of n stimuli in situations in which it is impractical (or sometimes even meaningless) to provide estimates

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for the stimuli. To this end, an expert (or a team of experts) provides pairwise comparison coefficients $a_{ij} > 0$, which are meant to be a substitute for the quotients s_i/s_j of the unknown (or even undefined) values of the stimuli $s_i, s_j > 0$. The quotients s_i/s_j are also sometimes called *relative weights* in the literature.

For the sake of our exposition we define an $n \times n$ pairwise comparison matrix simply as a square matrix $\mathbf{A} = [a_{ij}]$ such that $a_{ij} > 0$ for every $i, j = 1, \dots, n$. A pairwise comparison matrix \mathbf{A} is called *reciprocal* if $a_{ij} = \frac{1}{a_{ji}}$ for every $i, j = 1, \dots, n$ (then automatically $a_{ii} = 1$ for every $i = 1, \dots, n$). Let

$$\mathbf{A} = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{bmatrix}$$

where a_{ij} expresses an expert's relative preference of stimuli s_i over s_j .

A pairwise comparison matrix \mathbf{A} is called consistent if $a_{ij} \cdot a_{jk} = a_{ik}$ for every $i, j, k = 1, \dots, n$. While every consistent matrix is reciprocal, the converse is false in general.

Consistent matrices correspond to the ideal situation in which there are exact values s_1, \dots, s_n for the stimuli. The quotients $a_{ij} = s_i/s_j$ form a consistent matrix. Conversely, the starting point of the pairwise comparison inference theory is Saaty's theorem (see [9]) which states that for every $n \times n$ consistent matrix $\mathbf{A} = [a_{ij}]$ there exist positive real numbers s_1, \dots, s_n such that $a_{ij} = s_i/s_j$ for every $i, j = 1, \dots, n$. The vector $s = [s_1, \dots, s_n]$ is unique up to a multiplicative constant.

The challenge to the pairwise comparison method comes from the lack of consistency of the pairwise comparison matrices which arise in practice (while as a rule, all the pairwise comparison matrices are reciprocal). Given an $n \times n$ matrix \mathbf{A} which is not consistent, the theory attempts to provide a consistent $n \times n$ matrix \mathbf{C} which differs from matrix \mathbf{A} "as little as possible". One of the possible solutions to this problem was proposed by Saaty (see [9]). Let $s = [s_1, \dots, s_n]$ be the eigenvector of \mathbf{A} corresponding to σ , the largest eigenvalue in modulus of \mathbf{A} . By the Frobenius Theorem (see [5]), the eigenvalue σ is unique, positive, and simple. Furthermore, the vector s can be chosen with all components positive.

There has been an ongoing discussion about which method for finding solutions to a pairwise comparison matrix is better ([11, 2]). Amongst the strongest competitors are the Least Squares (*LS*), the Logarithmic Least Squares (*LLS*), the Geometric Means (*GM*), and the Eigenvector (*EV*) methods (See [6] and [11] for details). No decisive analytical proof has been published yet (to our knowledge). Furthermore, it is not clear whether or not an analytical proof can be devised (this issue is addressed in the conclusions of this paper). This seems to be a good reason for formulating the problem as an empirical experiment using a Monte Carlo approach.

Only two of the above methods will be considered in this paper, because of their practical importance:

- *GM* for simplicity (all one needs is a pocket calculator in most cases),
- *EV* for its claimed superiority ([11]) and its mathematical elegance which provides useful interpretations of the resulting weights.

We need to consider reciprocal matrices of orders 4 to 7. There is a proof that solutions to *R3* (*Rn* will stand for a reciprocal matrix of order n) by *GM* and *EV* are exactly the same;

and orders above seven are impractical due to the large number of comparisons required. There are two important problems to examine:

- How to measure the accuracy of solutions?
- How to generate, let us say, “*not – so – inconsistent*” (*NSI* for short) reciprocal matrices?

2 Comparison of solutions

Having a solution (obtained by any of the above methods) we can reconstruct the reciprocal matrix (by simple divisions of appropriate components of the solution vector s). The distance (e.g., Euclidean or Tchebychev) between the given and reconstructed matrices will be used as a measure of the accuracy of the solution.

Solutions of matrices are obtained by *GM* (geometric means, see [2]) and *EV* (eigenvector method, see [9]) and normalized to one (by dividing each component by the sum of all components). There is no way, however, of saying which solution (obtained by *GM* or *EV*) is better just by looking at them. For each vector of weights (a solution), $s_i, i = 1, \dots, n$, where all s_i are positive and $s_i \in [1/\max, \max]$ where \max is a scale maximum, we reconstruct the matrix $\mathbf{A} = [a_{ij}]$ where $a_{ij} = s_i/s_j$ for $i, j = 1, \dots, n$. The distance between the matrix \mathbf{A} and the original matrix is considered to be a measure of the solution accuracy. For a given fully consistent matrix this distance is equal to 0 (as expected), but for an inconsistent matrix it is greater than 0.

Two metrics have been applied:

- modified Euclidean, $\sqrt{\frac{\sum_{i,j=1}^n (a_{ij} - b_{ij})^2}{n^2}}$, (note the division by the total number of elements)
- Tchebychev (also known as the infinity metric) , $\max(|a_{ij} - b_{ij}|)$ for $i, j = 1, \dots, n$.

The division of the Euclidean distance by the number of matrix elements is necessary for comparing it with the Tchebychev distance which is the maximum distance between two corresponding matrix elements. Needless to say, this operation is just a simple scaling and has no influence on the final results.

No formal theorem has been proven (to our knowledge) showing that for a given matrix \mathbf{A} which is close enough to a consistent matrix \mathbf{C} , the solutions of \mathbf{C} are acceptable approximations for the problem described by \mathbf{A} . The inference that for a matrix with small inconsistency, the solutions of weights are close enough to real values of stimuli is purely speculative.

3 Generating *NSI* matrices

One should note that in itself the randomization of matrices is a difficult process but it does not matter if it is not absolutely perfect. From a statistical point of view, however, it is imperative that an identical set of randomly generated matrices is used for both methods since the main goal of this study is to compare the two methods. Any reasonably random set of matrices should suffice.

The problem of generating consistent reciprocal matrices in a random way is more complicated than it seems to be. The straight forward method of generating random matrices, checking their consistency factor (cf), and discarding those which are outside an assumed cf range is impractical. Our preliminary experiments have established that too many matrices (of the order of 10^9 or more) would need to be examined to find at least one matrix within an assumed range of inconsistencies. A more constructive approach needs to be implemented. A fully consistent reciprocal matrix is generated randomly. This matrix is then again randomized by multiplying the elements above the main diagonal by a randomizing multiplier (RM). The randomizing multiplier RM is constructed by adding or subtracting (at random) a randomly generated number $\rho \in [0, 1]$ which is multiplied by a given deviation (D):

$$RM = 1 \pm \rho \cdot D$$

For $D = 0$ we get fully consistent matrices. It is expected that the consistency factor should increase with increasing D . Two consistency factors are used in our analysis: eigenvalue-based (as introduced in [9]) and triad-based (as introduced in [7]).

The definition of consistency of a pairwise comparison matrix \mathbf{A} , based on eigenvalues, was introduced by Saaty [9]. His consistency definition is given by the following formula:

$$cf = \frac{\lambda_A - \text{order}(\mathbf{A})}{(\text{order}(\mathbf{A}) - 1)\lambda_{\text{random}}}$$

where λ_A is the largest eigenvalue of the reciprocal matrix \mathbf{A} and λ_{random} is the largest eigenvalue of randomly generated reciprocal matrices of the same order as matrix \mathbf{A} (see [9]). For a given order, λ_{random} is constant and many researchers do not use it. This factor is also omitted in our formula for cf but one may always divide our results by λ_{random} (published in [9]) to get the strict definition of Saaty's cf .

It is worthwhile to note that we are unable to establish an explicit analytical relationship between cf and D . In particular we do not know which D is necessary for a given cf (for either of the definitions). For each matrix generated for a chosen deviation D , we compute cf (according to each of the above mentioned definitions) and store it with the other results for further analysis.

4 Interpretation of results

In essence, solutions obtained by both methods, geometric means (GM) and eigenvector (EV), turned out to be close enough to the given input matrix. In this respect both methods are accurate enough for most practical applications. As the enclosed Table 1 demonstrates, the biggest difference between average deviations of GM and EV solutions is 0.00019 for the Euclidean metric and 0.00355 for the Tchebychev metric. For practical applications, this precision is far better than expected. After all we are talking, in most cases, about relative subjective comparisons and one tenth of a percent is usually below our threshold of perception.

<i>Ord</i>	<i>D</i>	<i>cf</i>		Euclidean metric			Tchebychev metric		
		<i>triad</i>	λ	<i>dist</i>	<i>diff</i>	<i>wins</i> ¹	<i>dist</i>	<i>diff</i>	<i>wins</i> ²
4	0.1	0.129	0.001	0.0153	0.00000	51.0%	0.1732	0.00002	56.5%
	0.2	0.238	0.003	0.0302	0.00001	52.3%	0.3420	0.00012	59.4%
	0.3	0.333	0.007	0.0451	0.00003	53.4%	0.5080	0.00037	61.1%
	0.4	0.416	0.014	0.0601	0.00008	54.6%	0.6744	0.00083	62.1%
	0.5	0.490	0.022	0.0755	0.00017	55.6%	0.8444	0.00149	62.7%
5	0.1	0.160	0.001	0.0138	0.00000	51.0%	0.2212	0.00003	52.7%
	0.2	0.293	0.004	0.0273	0.00001	52.2%	0.4356	0.00020	54.4%
	0.3	0.406	0.009	0.0407	0.00003	53.5%	0.6472	0.00061	55.7%
	0.4	0.502	0.017	0.0544	0.00009	55.0%	0.8592	0.00131	56.6%
	0.5	0.585	0.027	0.0685	0.00019	56.8%	1.0761	0.00236	57.2%
6	0.1	0.180	0.001	0.0122	0.00000	51.1%	0.2546	0.00003	51.5%
	0.2	0.328	0.004	0.0241	0.00001	52.4%	0.5017	0.00025	52.7%
	0.3	0.450	0.010	0.0361	0.00003	53.9%	0.7441	0.00082	53.8%
	0.4	0.553	0.019	0.0483	0.00007	55.7%	0.9884	0.00173	54.5%
	0.5	0.640	0.030	0.0610	0.00017	57.8%	1.2397	0.00300	55.1%
7	0.1	0.194	0.001	0.0108	0.00000	51.1%	0.2798	0.00004	51.1%
	0.2	0.351	0.005	0.0214	0.00001	52.5%	0.5505	0.00030	52.1%
	0.3	0.480	0.011	0.0321	0.00002	54.2%	0.8172	0.00097	52.8%
	0.4	0.587	0.020	0.0430	0.00006	56.2%	1.0850	0.00210	53.6%
	0.5	0.675	0.033	0.0544	0.00014	58.9%	1.3624	0.00355	54.2%

Table 1. Monte Carlo results for Euclidean and Tchebychev metrics (1,000,000 matrices were tested for each case of order and deviation)

Notations and abbreviations used in Table 1:

Ord matrix order
D deviation to perturb matrix elements
cf consistency factors
triad triad consistency factor
 λ eigenvalue-based consistency factor (computed without λ_{random} factor)
dist distance between the reconstructed and given matrices
*wins*¹ frequency of winning by *GM* over *EV* method in %
*wins*² frequency of winning by *EV* over *GM* method in %
diff difference in accuracy between the winning method and the other method

The implementation in C was fast. For each deviation *D*, 1,000,000 cases of *NSI* matrices were generated and analyzed in approximately 10 minutes on a personal computer with a 100MHz Pentium CPU. The results were checked against those produced by a prototype program written in APL2. The built-in operations on entire arrays and the operator “each” that applies a function itemwise to a vector of arrays permitted a simplified implementation that was not only shorter, but also easier to understand and modify. In fact, errors in the C code could easily have gone undetected were it not for the availability of the prototype.

Although execution of the APL2 program was slower, fewer matrices were needed in the calculations used for checking purposes. No attempt was made at optimising the APL2 program for speed since it was decided early on that it would serve primarily as a reference version on which different ideas could be tried easily and changes made quickly, whereas the C version would be used for the full calculations.

The values of the deviation D used were 0.1, 0.2, 0.3, 0.4, and 0.5. It is worthwhile noting that $D = 0.5$ may, in extreme cases, result in a matrix element that is 50 to 150 percent of the original element's value (that is a ratio of 1:3). The smallest deviation $D = 0.1$ perturbs an element by at most 10% which creates very small inconsistencies and has no practical influence on accuracy.

Table 1 also shows that the results generated by the *GM* method are better when the Euclidean metric is used while the *EV* method results are better for the Tchebychev metric! This should not be a surprise. After all, geometric means (*GM*) are “means”, and therefore the Euclidean metric (which “equalizes” differences of all elements) generates better results for *GM* than for *EV* solutions. On the other hand, the eigenvalue method (*EV*) finds the matrix with elements as close as possible to the given matrix which is reflected by the Tchebychev metric (the maximum of the absolute differences).

The programs for computing all the results are available to interested readers by email.

5 Conclusions

The lack of a clear cut answer as to which method generates more accurate solutions was a surprise, but the results are nevertheless of practical importance. For some applications one of methods may be better than the other. Simplicity and balanced distribution of estimations of all stimuli would dictate the use of the geometric means method while the more sophisticated computation of eigenvectors may contribute to a better precision on each individual component of the solution.

It is improbable that an analytical solution can be devised in a situation where the results favour *GM* over *EV* for one metric while favouring *EV* over *GM* for another metric. More importantly, statistical evidence of convergence to a solution has been observed. The theory of reciprocal matrices states that solutions for both methods (*GM* and *EV*) only have sense for fully consistent matrices. For inconsistent matrices no solution exists. In fact it cannot exist even if there are only three inconsistent judgements (that is one triad of inconsistent judgments; for details see [3]) since having a solution, we can reconstruct the relative judgments by simple division. This contradicts the assumption that they were inconsistent. Until now it has been tacitly assumed that for “*not – so – inconsistent*” matrices the obtained solutions should be close enough approximations but (to our knowledge) no formal theorem to this effect has been published.

Table 1 shows that the solutions for more consistent matrices are more precise (for either of the methods). This is not surprising, yet not a trivial observation, since a phenomena of rank reversal exists for extremely inconsistent matrices (see [11]). Rank reversal takes place where a solution by one method ranks stimuli differently from another method. A formal proof that “for a matrix with small enough inconsistency no rank reversal should take place” still challenges us.

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